5. The Flory approximation

Given our results this far, we can now embark in developing an approach which is able to give the sought for exponents, or at least an approximation, using excluded volume as a key ingredient.

First, let us write the exact probability for a chain of monomers with self-avoidance. Here we use, for the chain itself, a freely-jointed-chain because we know that local rigidity can be scaled away. Thus:

$$P_{SAW}(\{\vec{r}_i\}) = \frac{\prod_{i=1}^{N} p(\vec{r}_i) \cdot \overbrace{\prod_{i=0}^{N-1} \prod_{j=i+1}^{N} \Theta\left(|\vec{R}_i - \vec{R}_j| - 2r_0\right)}^{\text{guarantees no overlaps between any monomer pair}}}{Z_{SAW}}$$

self avoiding walk

Reminder: the Heaviside $\Theta(x)$ is

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

and we define here

r_0 = radius of a monomer

$\vec{R}_i$ = position of monomer i = $\sum_{l=1}^{i} \vec{r}_l$ (we place the 0th monomer at the origin)

The partition function Z_{SAW} is of course

$$Z_{SAW} = \int d\vec{r}_1 \ldots d\vec{r}_N \prod_{i=1}^{N} p(\vec{r}_i) \prod_{i=0}^{N-1} \prod_{j=i+1}^{N} \Theta\left(|\vec{R}_i - \vec{R}_j| - 2r_0\right)$$

The computation of $\langle \vec{R}_{ee}^{\,2} \rangle_{SAW}$ is then simply

$$\langle \vec{R}_{ee}^{\,2} \rangle = \int d\vec{r}_1 \dots d\vec{r}_N \, P_{SAW}(\{\vec{r}_i\}) \cdot \vec{R}_{ee}^{\,2}$$

$$\vec{R}_{ee} = \sum_{i=1}^{N} \vec{r}_i$$

The calculation is analytically intractable if we try any exact approach. [Of course, it can be done numerically].

What kind of approximation can we work out?

In the same way as we did for the FJC, can we obtain a probability for the end-to-end vector;

$$P_{SAW}(\vec{R}_{ee}) = \int d\vec{r}_1 \dots d\vec{r}_N \, P_{SAW}(\{\vec{r}_i\}) \, \delta(\vec{R}_{e} - \sum_i \vec{r}_i)$$

and then

$$\langle \vec{R}_{ee}^{\,2} \rangle = \int d\vec{R}_{ee} \, \vec{R}_{ee}^{\,2} \, P_{SAW}(\vec{R}_{ee})$$

We must thus work on $P_{SAW}(\vec{R}_{ee})$ and look for clever approximations.

Let's write again

$$P_{SAW}(\vec{R}_{ee}) = \frac{1}{Z_{SAW}} \int d\vec{r}_1 \dots d\vec{r}_N \prod_{i=1}^{N} p(\vec{r}_i) \overbrace{\prod_{i=0}^{N-1} \prod_{j=i+1}^{N} \Theta(|\vec{R}_i - \vec{R}_j| - 2b)}^{\substack{\text{survival probability} \\ S(\{\vec{r}_i\})}} \cdot$$

$$\cdot\, \delta(\vec{R}_{ee} - \sum_i \vec{r}_i)$$

We multiply and divide by

$$\int d\vec{r}_1 \dots d\vec{r}_N \prod_{i=1}^{N} p(\vec{r}_i)\, \delta(\vec{R}_{ee} - \sum_i \vec{r}_i) = P_{FJC}(\vec{R}_{ee})$$

and we obtain (still exact here):

$$P_{SAW}(\vec{R}_{ee}) = \frac{1}{Z_{SAW}} P_{FJC}(\vec{R}_{ee}) \underbrace{\int d\vec{r}_1 \dots d\vec{r}_N \underbrace{\frac{\prod_{i=1}^{N} p(\vec{r}_i)\, \delta(\vec{R}_e - \sum_i \vec{r}_i)}{P_{FJC}(\vec{R}_e)}}_{P_{FJC}(\{\vec{r}_i\} | \vec{R}_{ee})} S(\{\vec{r}_i\})}_{\langle S \rangle (\vec{R}_{ee})}$$

$P_{FJC}(\{\vec{r}_i\} | \vec{R}_{ee})$ is the conditional probability of a given conformation $\{\vec{r}_i\}$ given $\vec{R}_{ee}$, properly normalized

$\langle S \rangle (\vec{R}_{ee})$ is the average of the function S, for a given $\vec{R}_{ee}$

We can also rewrite Z_{SAW} :

$$Z_{SAW} = \int d\vec{r}_1 \dots d\vec{r}_N \prod_{i=1}^{N} p(\vec{r}_i) \, S(\{\vec{r}_i\}) =$$

$$= \int d\vec{R}_{ee} \int d\vec{r}_1 \dots d\vec{r}_N \prod_{i=1}^{N} p(\vec{r}_i) \, S(\{\vec{r}_i\}) \, \delta\left(\vec{R}_{ee} - \sum_{i=1}^{N} \vec{r}_i\right) =$$

inserted

$$\int d\vec{R}_{ee} \, \delta\left(\vec{R}_{ee} - \sum_{i=1}^{N} \vec{r}_i\right) = 1$$

$$= \int d\vec{R}_{ee} \, P_{FJC}(\vec{R}_{ee}) \int d\vec{r}_1 \dots d\vec{r}_N \frac{\prod_{i=1}^{N} p(\vec{r}_i) \, \delta\left(\vec{R}_{ee} - \sum_{i=1}^{N} \vec{r}_i\right)}{P_{FJC}(\vec{R}_{ee})} S(\{\vec{r}_i\}) =$$

$$= \int d\vec{R}_{ee} \, P_{FJC}(\vec{R}_{ee}) \langle S \rangle (\vec{R}_{ee})$$

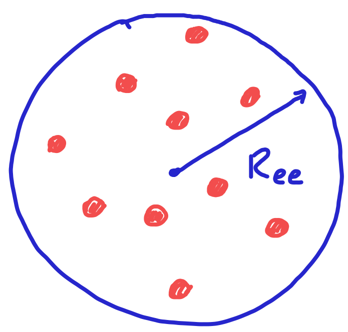
Putting everything together, we have at last the exact expression

$$P_{SAW}(\vec{R}_{ee}) = \frac{P_{FJC}(\vec{R}_{ee}) \langle S \rangle (\vec{R}_{ee})}{\int d\vec{R}_{ee} \, P_{FJC}(\vec{R}_{ee}) \langle S \rangle (\vec{R}_{ee})}$$

This expression is exact because we just wrote $\langle S \rangle (\vec{R}_{ee})$, which we are not able to write exactly.

Flory approximation = approximation for $\langle S \rangle (\vec{R}_{ee})$

Consider the $N+1$ monomer as a gas of disconnected particles within a sphere of radius $|\vec{R}_{ee}| \doteq R_{ee}$



What is the probability that placing them in the volume there will be no overlap?

$P_1 = 1$ ← probability of acceptance of the first monomer (empty sphere)

$P_2 = 1 - \dfrac{c(2r_0)^d}{V(R_{ee})}$ ← must subtract the volume of the overlap

forbidden volume : $c(2r_0)^d$

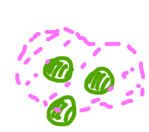
takes into account the angular part of the volume

volume of the sphere $= c R_{ee}^d$

same c

$= 1 - \dfrac{(2r_0)^d}{R_{ee}^d}$

$P_3 = 1 - 2\dfrac{(2r_0)^d}{R_{ee}^d}$ ← actually, we should also account for



→ three-body interactions

↓ assumption of the dilute gas

negligible

⋮

$P_k = 1 - (k-1)\dfrac{(2r_0)^d}{R_{ee}^d}$

What is the probability of succeeding at placing the N-1 monomers?

$$S \simeq \underbrace{p_1 \cdot p_2 \cdot p_3 \cdots p_{N+1}}_{\text{multiply because they must be verified at the same time}} = \prod_{k=1}^{N+1} p_k = e^{\sum_{k=1}^{N+1} \ln\left[1-(k-1)\frac{(2r_0)^d}{R_{ee}^d}\right]}$$

The dilute gas hypothesis implies: $\frac{(N+1)c(r_0)^d}{c\,R_{ee}^d} \ll 1$

$$\Rightarrow \quad \ln(1-x) \underset{x\ll 1}{\simeq} -x \qquad \text{and}$$

$$S \simeq e^{-\sum_{k=1}^{N+1}(k-1)\frac{(2r_0)^d}{R_{ee}^d}} = e^{-(2r_0)^d \frac{N(N+1)}{2R_{ee}^d}} \underset{N\gg 1}{\simeq} e^{-\underbrace{\frac{(2r_0)^d}{2}}_{B_2}\frac{N^2}{R_{ee}^d}}$$

Everything together gives

$$P_{\substack{SAW\\FLORY}}(\vec{R}_{ee}) = \frac{P_{FJC}(\vec{R}_{ee})\, e^{-B_2\frac{N^2}{R_{ee}^d}}}{\int d\vec{R}_{ee}\, P_{FJC}(\vec{R}_{ee})\, e^{-B_2 N^2/R_{ee}^d}} =$$

$$= \frac{e^{-\left[\frac{d\,R_{ee}^2}{2N} + B_2\frac{N^2}{R_{ee}^d}\right]} R_{ee}^{d-1}}{\underbrace{\int_0^\infty dR_{ee}\, R_{ee}^{d-1}\, e^{-\left[\frac{d\,R_{ee}^2}{2N} + B_2\frac{N^2}{R_{ee}^d}\right]}}_{Z_{\substack{SAW\\FLORY}} = \text{normalization}}}$$

already in spherical coordinates

the spherical part of the integral simplifies in the numerator and denominator

We must now compute $\langle \vec{R}_{ee}^{\,2} \rangle$:

$$\langle \vec{R}_{ee}^{\,2} \rangle = \frac{\displaystyle\int_0^\infty R_{ee}^{d-1} \, R_{ee}^{2} \, e^{-\left[\frac{d R_{ee}^2}{2N} + B_2 \frac{N^2}{R_{ee}^d}\right]} dR_{ee}}{\displaystyle\int_0^\infty R_{ee}^{d-1} \, e^{-\left[\frac{d R_{ee}^2}{2N} + B_2 \frac{N^2}{R_{ee}^d}\right]} dR_{ee}}$$

$\tilde{B}_2 = \frac{B_2}{b^d}$

Let's change variable : $x = \frac{R_{ee}}{b N^{\nu}}$ then

$$\langle \vec{R}_{ee}^{\,2} \rangle = b^2 N^{2\nu} \frac{\displaystyle\int_0^\infty x^{d-1} \, x^2 \, e^{-\left[N^{2\nu-1} \frac{d}{2} x^2 + N^{2-d\nu} \tilde{B}_2 \frac{1}{x^d}\right]} dx}{\displaystyle\int_0^\infty x^{d-1} \, e^{-\left[N^{2\nu-1} \frac{d}{2} x^2 + N^{2-d\nu} \tilde{B}_2 \frac{1}{x^d}\right]} dx}$$

as a first step we recognize that

$$x \, e^{-N^{2\nu-1} \frac{d}{2} x^2} = -\frac{1}{d} N^{1-2\nu} \frac{d}{dx} \left[e^{-N^{2\nu-1} \frac{d}{2} x^2} \right]$$

so we can integrate by parts the numerator :

$$b^2 N^{2\nu} \int_0^\infty x^{d-1} \, x \left\{ -\frac{1}{d} N^{1-2\nu} \frac{d}{dx} \left[e^{-N^{2\nu-1} \frac{d}{2} x^2} \right] \right\} e^{-N^{2-d\nu} \tilde{B}_2 \frac{1}{x^d}} dx =$$

$$= b^2 N^{2\nu} \Bigg\{ \underbrace{\left[-\frac{1}{d} N^{1-2\nu} \, e^{-N^{2\nu-1} \frac{d}{2} x^2} \, x^d \, e^{-N^{2-d\nu} \tilde{B}_2 \frac{1}{x^d}} \right]_0^\infty}_{0} +$$

$$+ \frac{1}{d} N^{1-2\nu} \int_0^\infty e^{-N^{2\nu-1} \frac{d}{2} x^2} \Big(d \, x^{d-1} \, e^{-N^{2-d\nu} \tilde{B}_2 \frac{1}{x^d}} +$$

$$+ d \, x^d N^{2-d\nu} \tilde{B}_2 \frac{1}{x^{d+1}} \, e^{-N^{2-d\nu} B_2 \frac{1}{x^d}} \Big) dx \Bigg\} =$$

$$= \overbrace{b^2 N \int_0^\infty x^{d-1} e^{-\left[N^{2\nu-1}\frac{d}{2}x^2 + N^{2-d\nu}\tilde{B}_2 \frac{1}{x^d}\right]} dx}^{=\text{ to the denominator}} +$$

$$+\tilde{B}_2 b^2 N^{3-d\nu} \int_0^\infty x^{d-1} e^{-\left[N^{2\nu-1}\frac{d}{2}x^2 + N^{2-d\nu}\tilde{B}_2 \frac{1}{x^d}\right]} \frac{1}{x^d}\, dx$$

Thus

$$\langle \vec{R}_{ee}^{\,2} \rangle = b^2 N + \tilde{B}_2 b^2 N^{3-d\nu} \frac{\int_0^\infty x^{d-1} e^{-\left[N^{2\nu-1}\frac{d}{2}x^2 + N^{2-d\nu}\tilde{B}_2 \frac{1}{x^d}\right]} \frac{1}{x^d} dx}{\int_0^\infty x^{d-1} e^{-\left[N^{2\nu-1}\frac{d}{2}x^2 + N^{2-d\nu}\tilde{B}_2 \frac{1}{x^d}\right]} dx}$$

$$= b^2 N + \tilde{B}_2 \left\langle \frac{1}{x^d} \right\rangle b^2 N^{2-(d\nu-1)}$$

If we compare this formula with our companion equation

$$\langle \vec{R}_{ee}^{\,2} \rangle \simeq b^2 N + C\, N^{2-\gamma}$$

we clearly identify $\gamma = d\nu - 1$

We must now identify $\left\langle \frac{1}{x^d} \right\rangle$. First we must find the good value of ν. The probability is

$$\frac{e^{-\left[\frac{d}{2} N^{2\nu-1} x^2 + \tilde{B}_2 N^{2-d\nu} \frac{1}{x^d}\right] + (d-1)\ln x}}{\int_0^\infty e^{-\left[\frac{d}{2} N^{2\nu-1} x^2 + \tilde{B}_2 N^{2-d\nu} \frac{1}{x^d}\right] + (d-1)\ln x} dx}$$

We cannot compute the average exactly, thus we use the saddle-point (also called "gaussian" approximation)

We look for the maximum of the probability : it corresponds to the minimum of

$$\frac{d}{2} N^{2\nu-1} x^2 + \tilde{B}_2 N^{2-d\nu} \frac{1}{x^d} - (d-1) \ln x$$

$$\Rightarrow \frac{d}{dx}\left[\frac{d}{2} N^{2\nu-1} x^2 + \tilde{B}_2 N^{2-d\nu} \frac{1}{x^d} - (d-1) \ln x\right] =$$

$$= d N^{2\nu-1} x - d \tilde{B}_2 N^{2-d\nu} \frac{1}{x^{d+1}} - (d-1) \frac{1}{x} = 0$$

$$\Rightarrow d N^{2\nu-1}\left[x - \tilde{B}_2 N^{3-(d+2)\nu} \frac{1}{x^{d+1}} - \frac{(d-1)}{d} N^{1-2\nu} \frac{1}{x}\right] = 0$$

We solve this equation in two cases :

1. $$x - B_2 N^{3-(d+2)\nu} \frac{1}{x^{d+1}} = 0$$

$$\Rightarrow \bar{x} = \left(\tilde{B}_2\right)^{1/d+2} N^{\frac{3}{d+2} - \nu}$$

2. $$x - \frac{(d-1)}{d} N^{1-2\nu} \frac{1}{x} = 0$$

$$\Rightarrow \bar{x} = \sqrt{\frac{d-1}{d}}\, N^{1/2 - \nu}$$

Case 1

choose $\nu = \frac{3}{d+2}$ (till now ν was arbitrary) and we obtain

$$\bar{x} = (B_2)^{1/d+2}$$

closer inspection of the equation to solve tells that

$$d\, N^{\frac{4-d}{d+2}} \left[x - \tilde{B}_2 \frac{1}{x^{d+1}} - \underbrace{\frac{(d-1)}{d} N^{\frac{d-4}{d+2}} \frac{1}{x}}_{\to 0 \text{ if } N \to \infty \text{ only if } d<4 \text{ otherwise it diverges}} \right] = 0$$

thus, solving case 1 is legitimate only if $d < 4$

In this case ($d<4$) we can in principle compute the leading correction

$$\bar{x} = (\tilde{B}_2)^{\frac{1}{d+2}} + c\, N^{-\beta} \qquad (\beta > 0)$$

$$\left[(\tilde{B}_2)^{1/d+2} + c\, N^{-\beta} \right] - \tilde{B}_2 \frac{1}{\left[(\tilde{B}_2)^{\frac{1}{d+2}} + c N^{-\beta} \right]^{d+1}} - \frac{d-1}{d} \frac{N^{\frac{d-4}{d+2}}}{(\tilde{B}_2)^{1/d+2} + c N^{-\beta}} = 0$$

$$\cancel{(\tilde{B}_2)^{\frac{1}{d+2}}} + c N^{-\beta} - \tilde{B}_2^{\frac{1}{d+2}} \left(\cancel{1} - \frac{c}{(\tilde{B}_2)^{1/d+2}} (d+1) N^{-\beta} \right) - \frac{d-1}{d} \frac{1}{(\tilde{B}_2)^{1/d+2}} N^{\frac{d-4}{d+2}} = 0$$

$$\Rightarrow \quad (d+2)\, c\, N^{-\beta} - \frac{d-1}{d} \frac{1}{(\tilde{B}_2)^{1/d+2}} N^{\frac{d-4}{d+2}} = 0$$

$$\Rightarrow \quad c = \frac{d-1}{d(d+2)} \frac{1}{(\tilde{B}_2)^{1/d+2}} \qquad \beta = \frac{4-d}{d+2}$$

We must also look at the second derivative in $\bar{x}$ (for sake of simplicity we take it at $\bar{x} = (\tilde{B}_2)^{1/d+2}$)

$$\frac{d^2}{dx^2}\left[-\frac{d}{2} N^{\frac{4-d}{d+2}} x^2 - \tilde{B}_2 N^{\frac{4-d}{d+2}} \frac{1}{x^d} + (d-1)\ln x\right]_{\bar{x}} =$$

$$= -d N^{\frac{4-d}{d+2}} - d(d+1) N^{\frac{4-d}{d+2}} - (d-1)(\tilde{B}_2)^{-2/d+2} =$$

$$= -d(d+2) N^{(4-d)/d+2} - (d-1)(\tilde{B}_2)^{-2/d+2} \underset{N\gg1}{\simeq} -(d+2)d\, N^{\frac{4-d}{d+2}}$$

If we approximate the exponential to the second order:

$$e^{g(x)} \simeq e^{g(\bar{x}) - \frac{1}{2} N^{\frac{4-d}{d+2}} d(d+2)(x-\bar{x})^2 + \dots}$$

$\frac{4-d}{d+2} > 0$ if $d<4$

$\frac{4-d}{d+2} < 0$ if $d>4$

We can thus draw

max moves toward $(\tilde{B}_2)^{1/d+2}$ and becomes steeper and steeper, and narrower and narrower around $\bar{x}$, if $d<4$, because it is normalised

$\Rightarrow$ it becomes akin to a Dirac-delta

$$\left\langle \frac{1}{x^d} \right\rangle \simeq \frac{1}{\bar{x}^d} = (\tilde{B}_2)^{-\frac{d}{d+2}}$$

we have thus

$$\langle \vec{R}_{ee}^{\,2} \rangle \simeq b^2 N + b^2 (\tilde{B}_2)^{\frac{2}{d+2}} N^{2\frac{3}{d+2}}$$

$$\Rightarrow \quad \nu_F = \frac{3}{d+2} \qquad d < 4$$

↳ Flory

If $d > 4$ the choice $\nu_F = \frac{3}{d+2}$ breaks apart because:

- the term $\frac{1}{x}$ is going to dominate
- the second derivative of the argument of the exponential goes to 0, and the distribution becomes broader.

Thus, if $d > 4$ we must use

Case 2

$$\bar{x} = \sqrt{\frac{d-1}{d}}\, N^{\frac{1}{2}-\nu} \qquad \text{and choose} \quad \nu = 1/2$$

$$\Rightarrow \quad \bar{x} = \sqrt{\frac{d-1}{d}}$$

In this case the equation is

$$d N^{2\nu-1} \bar{x} - d \tilde{B}_2 N^{2-d\nu} \frac{1}{\bar{x}^{d+1}} - (d-1)\frac{1}{\bar{x}} = 0$$

with $\nu = \frac{1}{2}$ becomes

$$d\,\bar{x} - \underline{d \tilde{B}_2 N^{2-d/2} \frac{1}{\bar{x}^{d+1}}} - (d-1)\frac{1}{\bar{x}} = 0$$

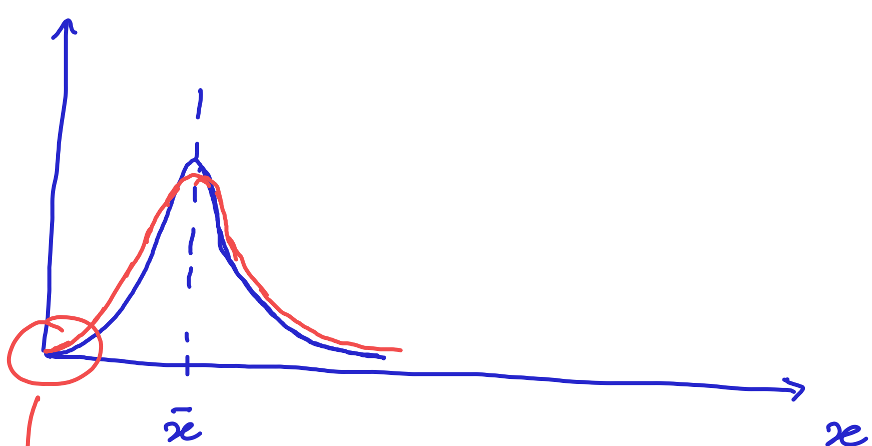
The term underscored in red is indeed progressively less important as N grows if $d > 4$, but would dominate for $d < 4$ (Case 1 !)

Again we could compute the corrections (not done here).

What about the second derivative in $\bar{x}$?

$$-d - (d-1)\frac{1}{\bar{x}^2} = -d - (d-1)\frac{d}{d-1} = -2d < 0 \quad \underline{\text{ok}}$$

In this case, as $N \to \infty$ we have a more delicate situation: graphically



even if the term $\frac{1}{x^d}$ does not dominate the solution, it dominates the behavior at $x \to 0$

Actually this is good, because

$$\left\langle \frac{1}{x^d} \right\rangle = \frac{\int_0^\infty \frac{1}{x^d}\, e^{-\left[\frac{d}{2}x^2 - (d-1)\ln x + \tilde{B}_2 N^{2-d/2}\frac{1}{x^d}\right]}\, dx}{\int_0^\infty e^{-\left[\frac{d}{2}x^2 - (d-1)\ln x + \tilde{B}_2 N^{2-d/2}\, 1/x^d\right]}\, dx}$$

this would be a non integrable divergence at $x \to 0$, which $e^{(d-1)\ln x} = x^{d-1}$ would not control ($\frac{1}{x}$ is also not integrable). The term e^{-1/x^d} controls it

We thus have, for $d>4$

$$\langle \tilde{R}_{ee}^{2} \rangle = b^2 N + b^2 \tilde{B}_2 \left\langle \frac{1}{x^d} \right\rangle N^{3-d/2}$$

but $3-d/2 < 1$ if $d>4$,
thus is negligible compared to the first term

$$\Rightarrow \quad \langle \tilde{R}_{ee}^{2} \rangle = b^2 N \quad \Rightarrow \quad \nu_F = 1/2$$

In conclusion, the Flory approximation gives

$$\nu_F = \frac{3}{d+2} = \begin{cases} 1 & \text{in } d=1 \\ \frac{3}{4} = 0.75 & d=2 \\ 0.6 & d=3 \end{cases}$$

(correct! once the direction is chosen, cannot fold back)

$\left.\begin{array}{l} d=2 \\ d=3 \end{array}\right\}$ agreement with experiments!

Agreement with the irrelevance of excluded volume for $d \geq 4$ as found in the two arguments

$\nu_F = 1/2$ if $d>4$ $\Longrightarrow$ $d \geq 4$

$\nu_F = 1/2$ if $d=4$ (agreement between the two formulas)

Side-note: exact result in $d=2$ is $\nu = \frac{3}{4}$!!!

best estimate in $d=3$, from analytical calculations (renormalisation group) and numerical approaches, is

$$\nu = 0.588\ldots$$

$\nu_F = 0.6$ is very good!!!

Textbook approach to Flory:

The argument of the exponential is

$$-\frac{d\,R_{ee}^2}{2b^2N} - B_2\frac{N^2}{R_{ee}^d}$$

Remembering that the probability of a given state Γ is

$$p(\Gamma) = \frac{1}{Z}\, e^{-\beta E(\Gamma)}$$

by analogy it is defined

$$F(R_{ee}) = k_BT\frac{d\,R_{ee}^2}{2b^2N} + k_BT\,B_2\frac{N^2}{R_{ee}^d}$$

this is the state

this is a free energy because it comes from the sum of many different microscopic states, all with the same R_{ee}

Equilibrium thermodynamics tells that the minimum of the free energy is the equilibrium state. Thus find the minimum of $F(R_{ee})$:

$$\frac{d\,k_BT}{b^2N}R_{ee} - k_BT\,d\,B_2\frac{N^2}{R_{ee}^{d+1}} = 0$$

$$\Rightarrow \quad \bar{R}_{ee} = \left(b^2B_2\right)^{1/d+2} N^{3/d+2}$$

which is the same result that we found.

Then, since for $d > 4$ it would be $\nu_F = \frac{3}{d+2} < \frac{1}{2}$

but since excluded volume leads to expansion of the FJC,

$\nu < \frac{1}{2}$ is inconsistent, and since qualitative arguments

tell that excluded volume is irrelevant above $d = 4$, we

conclude that Flory breaks down above $d = 4$.

We saw that this is wrong, and a careful treatment

of Flory makes it fully consistent.